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Dynamic Aspects of the Undulation Instability in Smectic A Liquid Crystals

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The hydrodynamic behavior of a smectic A dilated perpendicularly to the layers is described by a set of partial differential equations. We study the dynamics of the undulation instability in terms of a characteristic equation of sixth degree. For the undulation to develop at a finite growth rate the dilation must exceed a positive threshold value. Above threshold we study the range of undamped modes. The fastest growing one is dominant and its wave vector corresponds to the static threshold. We conclude by some qualitative conjectures about the effects of a laminar flow, the most important being the breaking of symmetry.

INTRODUCTION

When a smectic A liquid crystal is subjected to a dilative strain perpendicular to the layers it displays a characteristic instability against undulation of the layers.[1–7] When the dilation exceeds a certain threshold value the stress is relaxed by an alternating tilt of the layers. This is manifested by characteristic patterns that can be observed directly or by light scattering experiments. A further increase of the dilative strain results in defect patterns which, however, tend to inherit the features of the initial undulation.[8,9]

This paper is an outgrowth of a research program on the undulation instability in smectic A liquid crystals in the presence of laminar flow parallel to the layers. In this framework, we found it useful to undertake a

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deeper study of the undulation modes and their growth rates using a linear approximation somewhat more elaborate than usual, which allows, among other things, to identify explicitly the long wave mode. Simpler treatments can be found in Refs. 1, 2, 5, 13, and 14.

This paper is organized as follows: in Section 1, we define the sample geometry and summarize the fundamental notions of smectic elasticity and hydrodynamics. In Section 2 we formulate the problem in terms of a set of partial differential equations. In Section 3 we study the Fourier-expanded solution by means of the associated characteristic equation of sixth degree. In Section 4 we calculate the growth rates of the modes as functions of their wave vectors and find the fastest growing mode. In Section 5, we point out the modification brought about by laminar flow and draw some qualitative conclusions.

1. FUNDAMENTAL NOTIONS

Throughout this paper, we adopt for the geometry of the sample, the following conventions (Figure 1): the smectic is sandwiched between two plane parallel plates, assumed infinite, a distance $2d$ apart. The unperturbed smectic layers are parallel to the plates. Equivalently, the long molecular axes are on the average perpendicular to the plates. We call this orientation homeotropic; some authors (see Ref. 8) prefer the designation perpendicular.

We choose a righthanded, orthogonal cartesian coordinate system with origin O in the midplane between the plates, the x and y axes parallel, the z axis perpendicular to the plates. Time will be denoted t.

We assume that the smectic sample is strongly anchored, that is to say, that the orientation at the plates stays homeotropic. The flow of a smectic is quite generally described by a velocity field $v_i = v_i(x_j,t)$ satisfying the equations of Navier-Stokes:[10–14]

$$\rho \dot{v}_i = -\partial_i P + \partial_j \sigma_{ji} + G_i, \qquad (i,j = x,y,z) \tag{1.1}$$

where ρ is the mass density, P the pressure, σ_{ji} the viscous stress tensor and G_i a volume force density, with obvious notations for derivatives and summation convention.

The flow is assumed to be incompressible, thus

$$\partial_i v_i = 0 \tag{1.2}$$

The viscous stress tensor σ_{ij} is given by[12,13]

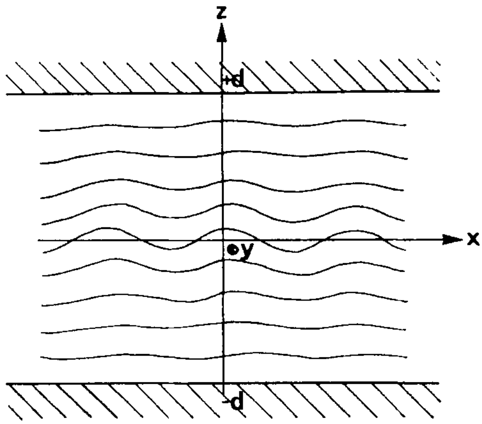

FIGURE 1 Sample geometry.

$$\sigma_{zz} = 2\eta_1 v_{zz}, \qquad \sigma_{zx} = 2\eta_3 v_{zx}, \qquad \sigma_{zy} = 2\eta_3 v_{zy}$$
$$\sigma_{xx} = 2\eta_2 v_{xx}, \qquad \sigma_{xy} = 2\eta_2 v_{xy}, \qquad \sigma_{yy} = 2\eta_2 v_{yy} \tag{1.3}$$

where η_1, η_2, η_3 are viscosities and

$$v_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i) \tag{1.4}$$

is the symmetrized velocity gradient tensor. For simplicity, the viscosity matrix is taken to be diagonal, so that, for instance, σ_{xx} does not depend on v_{zz}. For convenience, we will later even arbitrarily set $\eta_1 = \eta_2 = \eta_3 = \eta$.

The elasticity of smectics has been quite extensively discussed in the literature.[1,2,15–17] We shall quote here only the essentials.

A smectic displays elasticity of two kinds:

(1) bulk stiffness against compression or dilation normal to the layers expressed by an elastic modulus B; and

(2) orientational stiffness against splay and bend expressed by the Frank elastic moduli K_1 and K_3, respectively, while twist is assumed to be forbidden.

The distortion of the layer will, conventionally, be described in terms of a displacement field $u_i = \delta_{iz} u(x_j)$ in the z-direction normal to the layers.

The elastic free energy density F may, in a conventional approximation, be represented by

$$F = \frac{1}{2}B[u_z - \frac{1}{2}(u_x^2 + u_y^2)]^2 + \frac{1}{2}K_1[u_{xx} + u_{yy}]^2 + \frac{1}{2}K_3[u_{xz} + u_{yz}]^2 \tag{1.5}$$

where the subscripts denote partial derivatives.

The volume force density G_i is the conjugate of the layer displacement u_i represented by the functional derivative

$$G_i = -\frac{\delta F}{\delta u_i} = -\delta_{iz}\frac{\delta F}{\delta u} = \delta_{iz} G \tag{1.6}$$

Thus, it has naturally only a single component G in the z-direction. This can be explicitly written as:

$$\begin{aligned} G/B = u_{zz} &- \lambda_1^2(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) - \lambda_3^2(u_{xxzz} + u_{yyzz}) \\ &- u_z(u_{xx} + u_{yy}) - 2(u_x u_{xz} + u_y u_{yz}) + \frac{3}{2}(u_x^2 u_{xx} + u_y^2 u_{yy}) \\ &+ \frac{1}{2}(u_x^2 u_{yy} + u_y^2 u_{xx}) + 2u_x u_y u_{xy} \end{aligned} \tag{1.7}$$

where λ_1 and λ_2 are characteristic lengths defined by:

$$\lambda_1^2 = K_1/B \qquad \lambda_3^2 = K_3/B \tag{1.8}$$

in Eq. (1.7), the term containing λ_3, which corresponds to bend, is usually neglected, although it does not entail mathematical difficulties; of course, it only slightly modifies the final results.[5] In what follows, bend will be neglected and the subscript will be dropped from $K \equiv K_1$ and $\lambda \equiv \lambda_1$.

The equation of motion for the layer displacement, sometimes called permeation equation is[12]

$$\frac{du}{dt} - v_z = \zeta G \tag{1.9}$$

where ζ is the permeability also called permeation coefficient,[18] the equation describes momentum transfer by permeation.

2. FORMULATION OF THE PROBLEM

We start with Eqs. (1.1). Confined to two dimensions in the (xz)-plane they are explicitly:

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + 2\eta_2\frac{\partial^2 v_x}{\partial x^2} + \eta_3\frac{\partial^2 v_z}{\partial x\,\partial z} + \eta_3\frac{\partial^2 v_x}{\partial z^2} \tag{2.1}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_z\frac{\partial v_z}{\partial z}\right) =$$
$$-\frac{\partial P}{\partial z} + \eta_3\frac{\partial^2 v_z}{\partial x^2} + \eta_3\frac{\partial^2 v_x}{\partial x\,\partial z} + 2\eta_1\frac{\partial^2 v_z}{\partial z^2} + G \qquad (2.2)$$

To eliminate the pressure terms we take cross derivatives and substract Eq. (2.2) from Eq. (2.1) to obtain

$$\rho\left[\frac{\partial}{\partial t}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + v_x\frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + v_z\frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\right.$$
$$\left. + \left(\frac{\partial v_x}{\partial x} - \frac{\partial v_z}{\partial z}\right)\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\right] = \eta_3\left(\frac{\partial^3 v_z}{\partial x^3} - \frac{\partial^3 v_x}{\partial z^3}\right)$$
$$+ (2\eta_1 - \eta_3)\frac{\partial^3 v_z}{\partial x\,\partial z^2} - (2\eta_2 - \eta_3)\frac{\partial^3 v_x}{\partial z\,\partial x^2} + \frac{\partial G}{\partial x} \qquad (2.3)$$

It will be noticed that the entire left hand side of Eq. (2.3) depends on the curl of the velocity and vanishes for irrotational flow. The last term on the left hand side vanishes identically for any incompressible flow.

We define the vorticity or strictly speaking, its negative y-component

$$\Xi = \frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \qquad (2.4)$$

We also introduce a stream function Ψ such that:

$$v_x = \frac{\partial \Psi}{\partial z} \qquad v_z = -\frac{\partial \Psi}{\partial x} \qquad (2.5)$$

Thereby the incompressibility condition in Eq. (1.2) is identically satisfied. More generally, one would introduce a vector potential.

With these definitions, Eq. (2.3) can be written compactly as

$$\rho\frac{d\Xi}{dt} = \frac{\partial G}{\partial x} - \eta_3\left[\frac{\partial^4\Psi}{\partial x^4} + 2\frac{\eta_1 + \eta_2 - \eta_3}{\eta_3}\frac{\partial^4\Psi}{\partial x^2\partial z^2} + \frac{\partial^4\Psi}{\partial z^4}\right] \qquad (2.6)$$

We see that if

$$\eta_1 + \eta_2 \approx 2\eta_3 \qquad (2.7)$$

then, to a good approximation, all viscosity coefficients may be replaced by η_3.

Therefore, we introduce a "single viscosity approximation" and arbitrarily set

$$\eta_1 = \eta_2 = \eta_3 = \eta \qquad (2.8)$$

We shall not pursue this line of thought any further. Instead, we prefer to return to Eq. (2.3) and establish a criterion for the importance of permeation in the present problem. With Eqs. (2.4) and (2.8), Eq. (2.3) becomes

$$\rho\frac{d\Xi}{dt} = \frac{\partial G}{\partial x} + \eta\left(\frac{\partial^2\Xi}{\partial x^2} + \frac{\partial^2\Xi}{\partial z^2}\right) \tag{2.9}$$

This is a diffusion equation for the vorticity where $\partial G/\partial x$ plays the role of a source.

On the right hand side, we substitute for v_z from the permeation Eq. (1.9) and collect terms containing $\partial G/\partial x$

$$\left[1 - \zeta\eta\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\right]\frac{\partial G}{\partial x} \tag{2.10}$$

We have to compare the two terms in the square brackets. We are thus led to define a permeation number

$$Pm = \frac{\zeta\eta}{\lambda d} = \frac{\zeta\nu\rho}{d}\sqrt{\frac{B}{K}} \tag{2.11}$$

and replace Eq. (2.10) by

$$[1 - Pm]\frac{\partial G}{\partial x} \tag{2.12}$$

Here $\nu = \eta/\rho$ is the kinematic viscosity, $(\lambda d)^{1/2}$ is a characteristic wavelength of the undulation pattern (see Ref. 5 or Section 3 of this paper). It is noteworthy that the permeation number Pm contains all relevant physical parameters of the smectic. Under usual circumstances $\zeta \sim 10^{-16}$ m^5/Js, $\eta \sim 10^{-1}$ Js/m^3, $\lambda \sim 10^{-9}$ m, $d \sim 10^{-4}$ m, so that $Pm \lesssim 10^{-4} \ll 1$. Permeation in the bulk is thus completely negligible. Consequently, we shall henceforth drop the second term in Eq. (2.10).

Permeation is, of course, important within a permeation boundary layer, such as introduced by de Gennes[13] and the Orsay Group,[3] and recently studied by Bartolino and Durand.[14] According to these authors, the thickness of the permeation layer is of the order of $(\zeta\eta)^{1/4}d^{1/2} \lesssim 10^{-6}$ m, which is certainly macroscopic and may often be significant. Here, however, we are interested in the bulk where the undulation takes place. We shall have more to say on this subject at the end of Section 4.

It is clearly possible to have a laminar flow parallel to the layers with a superposed homogeneous dilation or compression perpendicular to the layers. Indeed, Eq. (2.3) has the obvious solution

$$u_o(x, y, z, t) = \alpha z, \tag{2.13}$$

$$\mathbf{v}_o(x, y, z, t) = (V, 0, 0), \tag{2.14}$$

where α is a constant dilative or compressive strain and

$$V = a_0 + a_1 z + a_2 z^2. \tag{2.15}$$

The constants a_0, a_1, a_2 are to be determined from the boundary conditions; a_0 represents, of course, a trivial rigid motion.

It is natural to seek a more general solution in the form

$$u = u_0 + u_1, \tag{2.16}$$

$$\mathbf{v} = \mathbf{v}_o + \mathbf{v}_1, \tag{2.17}$$

where u_1, $\mathbf{v}_1$ are small perturbations. We substitute this into the basic Eqs. (2.3) and (1.2, 7, 9) which we then properly linearize.

Eventually, the problem is reduced to solving the set of partial differential equations:

$$\left[\rho\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right) - \eta\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\right]\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z},\right) = \frac{\partial G}{\partial x}, \tag{2.18}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0, \tag{2.19}$$

$$v_z = \frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x}, \tag{2.20}$$

$$G = B\left(\frac{\partial^2 u}{\partial z^2} - \alpha\frac{\partial^2 u}{\partial x^2} - \lambda^2\frac{\partial^4 u}{\partial x^4}\right), \tag{2.21}$$

with suitable boundary conditions. Explicitly we need only to specify the imposed dilation

$$u(z = \pm d) = \pm \alpha d \tag{2.22}$$

3. CHARACTERISTIC EQUATION

It is well known that if the dilative strain α surpasses a certain critical (or threshold) value α_c the smectic layers develop undulations to relax the stress and lower the total free energy (see Ref 5). We wish to study how the undulation pattern evolves in time. We start with the simple case when no flow is imposed, i.e., when $V = 0$.

The undulation, superposed upon the homogeneous dilation in Eq. (2.19), can be quite generally represented by a Fourier expansion

$$u(x, y, z, t) = \alpha z + \mathrm{S}\, u(q_x, q_y \mid z) \exp[i(q_x x + q_y y - \omega(q_x, q_y) t] + \text{c.c.}, \tag{3.1}$$

where, in an obvious way, $u(q_x, q_y \mid z)$ are Fourier coefficients, q_x, q_y are the plane wave vector components, and $\omega(q_x, q_y)$ is the corresponding frequency (real or complex), all these to be determined from the boundary conditions. At this stage, the nature of the sum S is unimportant, it does turn out to be a discrete series.

It is sufficient to represent the expansion by a single Fourier term and choose the wave vector to be in the x-direction. Thus, our ansatz is

$$u(x, z; t) = \alpha z + u(z) \exp[i(qx - \omega t)] + \text{c.c.}, \tag{3.2}$$

where

$$u \equiv u(z) = u(q, 0; z) \tag{3.3}$$

is the amplitude of the undulation and the subscript has been dropped from q_x.

Straightforward substitution into Eqs. (2.19–21) then yields the amplitudes

$$v_x \equiv v_x(z) = \frac{\omega}{q}\frac{du}{dz} \tag{3.4}$$

$$v_z \equiv v_z(z) = -i\omega u \tag{3.5}$$

$$G \equiv G(z) = B\left(\frac{d^2u}{dz^2} + \alpha q^2 u - \lambda^2 q^4 u\right). \tag{3.6}$$

Equation (2.18) becomes, in turn

$$i\omega T \frac{1}{q^4}\frac{d^4u}{dz^4} + \left[1 - \left(\frac{\omega}{Cq}\right)^2 - 2i\omega T\right]\frac{1}{q^2}\frac{d^2u}{dz^2} - \left[(\lambda q)^2 - \left(\frac{\omega}{Cq}\right)^2 - \alpha - i\omega T\right]u = 0 \tag{3.7}$$

where

$$T = \frac{\eta}{B} \tag{3.8}$$

is a characteristic relaxation time of the order of 10^{-7} s, while

$$C = \sqrt{\frac{B}{\rho}} \tag{3.9}$$

is the speed of smectic second sound of the order of 10 m/s.

At this point, let us assume that the function $u(z)$ has a well-defined parity and define the manifestly positive integrals

$$I_n = \int_{-d}^{+d} dz |u^{(n)}|^2. \tag{3.10}$$

We multiply Eq. (3.7) by the complex conjugate of u, and integrate across the layer. In view of the boundary conditions of Eq. (2.22) integration by parts yields the result:

$$i\omega T \frac{I_2}{q^4} - \left[1 - \left(\frac{\omega}{Cq}\right)^2 - 2i\omega T\right] \frac{I_1}{q^2} - \left[(\lambda q)^2 - \left(\frac{\omega}{Cq}\right)^2 - \alpha - i\omega T\right] I_0 = 0. \tag{3.11}$$

Now we want to establish whether or not the undulation propagates as a wave. We assume the wave to be unattenuated, so we take q to be real. As for ω, we decompose it into its real and imaginary parts, ω' and ω'', respectively

$$\omega = \omega' + i\omega''. \tag{3.12}$$

Then, the real and imaginary parts of Eq. (3.11) become, respectively

$$-\omega'' \frac{I_2}{q^4} - \left[1 - \frac{\omega'^2 - \omega''^2}{(Cq)^2} + 2\omega'' T\right] \frac{I_1}{q^2} - \left[(\lambda q)^2 - \frac{\omega'^2 - \omega''^2}{(Cq)^2} - \alpha + \omega'' T\right] I_0 = 0, \tag{3.13}$$

$$\omega' T \left\{ \frac{I_2}{q^4} + 2\left[1 + \frac{\rho\omega''}{\eta q^2}\right] \frac{I_1}{q^2} + \left[1 + 2\frac{\rho\omega''}{\eta q^2}\right] I_0 \right\} = 0. \tag{3.14}$$

One way to satisfy Eq. (3.14) is to have

$$\frac{2\rho\omega''}{\eta q^2} = -\frac{I_0 + 2I_1/q^2 + I_2/q^4}{I_0 + I_1/q^2} < 0, \tag{3.15}$$

hence, to leading order

$$\omega'' = -\frac{\eta q^2}{2\rho} \tag{3.16}$$

This corresponds to a rapidly decaying fluctuation. In fact, with the right values for q to be determined later, we find $|\omega''| \sim 10^8\ \mathrm{s}^{-1}$. A straightforward calculation also shows that, to leading order $\omega' \approx \omega''$.

The other, more interesting possibility to satisfy Eq. (3.14) arises for a vanishing real part of the frequency

$$\omega' = 0. \tag{3.17}$$

This corresponds to a stationary, nonpropagating undulation.

The preceding reasoning crucially depends on $T = \eta/B$ being different from zero in Eq. (3.14). Thus we have shown that it is the viscosity that prevents the existence of propagating solutions with well-defined parity.

It is now justified to set

$$u(z) = u_n \exp(i k_n z) + \text{c.c.}, \tag{3.18}$$

where u_n is a constant amplitude; the boundary conditions of Eq. (2.13) imply for the z-component of the wave vector:

$$k = k_n = \frac{n\pi}{2d}, \qquad n = 1, 2, \ldots \tag{3.19}$$

(the notation conforms to the generally accepted bad habit).

We confine the discussion to the fundamental mode ($n = 1$) which, of course, dominates. Accordingly we will henceforth assume

$$u(z) = u_1 \cos(kz), \qquad k = \pi/2d. \tag{3.20}$$

Substitution into Eq. (3.7) then yields, for fixed α, ω and k, a characteristic equation for q:

$$\lambda^2 q^6 - [\alpha + i\omega T]q^4 - [(\omega/C)^2 - (1 - 2i\omega T)k^2]q^2 - [(\omega/C)^2 + i\omega T k^2]k^2 = 0. \tag{3.21}$$

For the static case, $\omega = 0$, we then find

$$q^2(\lambda^2 q^4 - \alpha q^2 + k^2) = 0 \tag{3.22}$$

Hence, either

$$q_0^2 = 0 \tag{3.23}$$

independent of λ, α and k. This corresponds to the trivial solution $u(z) = 0$. Alternatively, we have the characteristic biquadratic equation

$$\lambda^2 q^4 - \alpha q^2 + k^2 = 0 \tag{3.24}$$

well known from the elementary treatment of the problem. The solutions to Eq. (3.24) are

$$q_\pm^2 = \frac{\alpha \pm \sqrt{\alpha^2 - 4\lambda^2 k^2}}{2\lambda^2}. \tag{3.25}$$

The condition that q^2 be real yields a positive threshold value α_c for the dilation:

$$\alpha \geqslant \alpha_c = 2\lambda k > 0 \tag{3.26}$$

Hence, we obtain for the wave vector at threshold

$$q_c^2 = q_+^2 = q_-^2 = k/\lambda. \tag{3.27}$$

Since, with increasing dilation α, q_-^2 asymptotically tends to zero, the interesting solution is q_+^2. Under usual circumstances, the orders of magnitude are $k \sim 10^4\ \mathrm{m}^{-1}$, $q \sim 10^6\ \mathrm{m}^{-1}$, thus $k^2 \ll q^2$.

Consequently, it is interesting to study the case of an "infinitely" thick sample, $k \to 0$. For non-decaying solutions we may assume, according to Eqs. (3.16 and 17), $\omega = i\omega''$. Then Eq. (3.21) simplifies to

$$\lambda^2 q^4 - (\alpha - \omega'' T)q^2 + (\omega''/C)^2 = 0, \tag{3.28}$$

where the trivial solution, $q_0^2 = 0$, has already been separated. The solutions are

$$q_\pm^2 = \frac{\alpha - \omega'' T \pm \sqrt{(\alpha - \omega'' T)^2 - (2\lambda\omega''/C)^2}}{2\lambda^2}. \tag{3.29}$$

In order for the undulation to be neither damped ($\omega'' \geqslant 0$), nor attenuated ($q^2 > 0$) the dilation must exceed a positive threshold value $\alpha_c(\omega'')$

$$\alpha \geqslant \alpha_c(\omega'') = (T + 2\lambda/C)\omega'' \geqslant 0 \tag{3.30}$$

This is an interesting result. It shows that the threshold dilation is proportional to the growth rate. Since $(\lambda/C) \ll T$, it follows that the threshold dilation is governed by and increases with the viscosity.

We return now to the full Eq. (3.21) for the case of a standing undulation ($\omega' = 0$) in a sample of finite thickness ($k \neq 0$):

$$\lambda^2 q^6 - [\alpha - \omega'' T]q^4 + [(\omega''/C)^2 + (1 + 2\omega'' T)k^2]q^2 + [(\omega''/C)^2 + \omega'' T k^2]k^2 = 0. \tag{3.31}$$

We separate first the long-wave solution q_0^2. We assume $q_0^2 < k^2$; consequently, we neglect the first two terms in Eq. (3.31), so that

$$q_0^2 \approx -\frac{(\omega''/C)^2 + \omega'' T k^2}{(\omega''/C)^2 + (1 + 2\omega'' T)k^2} k^2. \tag{3.32}$$

For a growing solution ($\omega'' > 0$) this is negative, corresponding to a spatially attenuated undulation. Again, only the trivial solution ($\omega'' = 0$, $q_0^2 = 0$) survives.

For the short-wave solutions ($q^2 \gg k^2$) we may neglect the last term in Eq. (3.31) and wind up with a quadratic equation for q^2

$$\lambda^2 q^4 - [\alpha - \omega'' T]q^2 + [(\omega''/C)^2 + (1 + 2\omega'' T)k^2] = 0 \tag{3.33}$$

which has the solutions

$$q_{\pm}^2 = \frac{\alpha - \omega''T \pm \sqrt{(\alpha - \omega''T)^2 - 4\lambda^2[(\omega''/C)^2 + (1 + 2\omega''T)k^2]}}{2\lambda^2}. \tag{3.34}$$

Again, we find a positive threshold dilation exceeding the static value

$$\alpha_c(\omega'') = 2\lambda k\sqrt{1 + 2\omega''T + (\omega''/Ck)^2} > 2\lambda k. \tag{3.35}$$

We could have investigated directly the solutions of the cubic Eq. (3.31) in terms of its proper discriminant. The procedure is, however, unduly cumbersome while yielding essentially the same results.

4. GROWTH RATES

To obtain a dispersion relation, we rewrite Eq. (3.31) as a quadratic equation for the growth rate ω''

$$\omega''^2 + \frac{\eta}{\rho}(q^2 + k^2)\omega'' + \frac{(\lambda^2 q^4 - \alpha q^2 + k^2)q^2}{q^2 + k^2} = 0 \tag{4.1}$$

The solution is best represented in terms of dimensionless quantities:

$$W_{\pm} = \frac{Q + b}{2}\left(-1 \pm \sqrt{1 - 4c\frac{Q(Q^2 - aQ + 1)}{(Q + b)^3}}\right), \tag{4.2}$$

where

$$a = \alpha/\lambda k, \tag{4.3}$$

$$b = \lambda k, \tag{4.4}$$

$$c = \beta\rho\lambda^2/\eta^2 = \rho K/\eta^2, \tag{4.5}$$

$$Q = \lambda q^2/k, \tag{4.6}$$

and

$$W = \frac{\rho\lambda}{\eta k}\omega''. \tag{4.7}$$

We are interested in finding the range of Q for which W is real and positive and the value $Q_{\max}$ for which W has a maximum.

To investigate Eq. (4.2) we look first at the second term under the square root. This is obviously small compared to unity in all interesting cases since: (a) for long waves $Q \to 0$, (b) for short waves $Q \to \infty$, (c) around static equilibrium $(Q^2 - aQ + 1) \to 0$. Consequently, to an excellent approximation, the solutions may be rewritten as:

$$W_+ = -c\frac{Q(Q^2 - aQ + 1)}{(Q + b)^2} \tag{4.8}$$

and

$$W_- = -(Q + b). \tag{4.9}$$

The latter is the same as

$$\omega''_- = -\frac{\eta}{\rho}(q^2 + k^2) \tag{4.10}$$

which corresponds to the rapidly decaying fluctuations of Eq. (3.16).

We look now at W_+. Obviously, $Q = 0$, $W_+ = 0$ is the trivial solution. For very long waves we have $0 < Q \ll 1$, hence W_+ is clearly negative and the solutions decay.

Thus we may focus attention to the range of short waves; i.e., $Q \gg b$. In this case we may replace Eq. (4.8) by

$$W_+ = -c\left(Q - a + \frac{1}{Q}\right)\left(1 - 2\frac{b}{Q}\right). \tag{4.11}$$

For physically interesting values of the parameters there are two intervals of Q where W_+ is nonnegative. The first is

$$0 \leqslant Q \leqslant 2b \tag{4.12}$$

which brings us back to long waves. Thus the pertinent range is

$$Q_- \leqslant Q \leqslant Q_+, \tag{4.13}$$

where Q_- and Q_+ are the solutions of the equation

$$Q^2 - aQ + 1 = 0, \tag{4.14}$$

while $a \geqslant 2$. This, of course, reproduces the elementary theory.

The maximum of $W_+(Q)$ occurs, to first order at

$$Q_{\max} = 1 + ab = 1 + \alpha, \tag{4.15}$$

that is to say:

$$q^2_{\max} = (1 + \alpha)k/\lambda \approx k/\lambda = q_c^2. \tag{4.16}$$

The corresponding fastest growth rate is

$$\omega''_{\max} = \frac{B}{\eta}(\alpha - 2\lambda k)(1 + 2\lambda k) \approx (\alpha - 2\lambda k)/T. \tag{4.17}$$

We find the threshold dilation by setting $\omega^{\parallel}_{\max} = 0$; thus we get

$$\alpha_c = 2\lambda k \tag{4.18}$$

as expected.

We conclude that the fastest growing and hence dominant undulation mode always occurs for the threshold value of the wave vector $q_c = (k/\lambda)^{1/2}$ with only an insignificant correction of the relative order of α.

As long as the regime can be approximately considered as linear, the dilation α cannot exceed a few λk. Hence we find for the undulation typical risetimes τ of the order of a few milliseconds

$$\tau = \frac{T}{\lambda k} = \frac{\eta d}{\sqrt{BK}} \sim 10^{-3}\ \mathrm{s} \tag{4.19}$$

Of course, it must be emphasized that very soon after an undulation has started, nonlinear effects take over and become all important. There are at least two effects. In the first place, the growth of the undulation is limited by saturation. Secondly, at a sufficiently strong dilation, the undulation pattern may develop into a defect texture.

We now return to the question of permeation. Near the plates, permeation certainly must occur, since otherwise the boundary conditions $v_x(z = \pm d) = 0$ cannot be met. For a layer of thickness δ let us define a characteristic permeation time

$$\tau_p(\delta) = \frac{\delta^2}{\zeta B}. \tag{4.20}$$

Permeative flow across a layer of thickness δ will compete with hydroelastic flow (i.e., the motion of the smectic layers, see Ref. 14) if τ_p is of the order of the risetime τ or less. Bulk permeation is associated with a time $\tau_p(\delta = d) \simeq 10^2$ s, very long compared to the risetime $\tau \simeq 10^{-3}$ s. We again conclude that bulk permeation is utterly negligible.

On the other hand, setting $\tau_p = \tau$, we find

$$\delta^2 = \zeta B\tau = \zeta\eta d\sqrt{B/K} = Pm \cdot d^2, \tag{4.21}$$

whence again $\delta \lesssim 10^{-6}$ m. This also corroborates our choice of the permeation number.

Another, perhaps more important reason to neglect permeation as such is that a much faster relaxation mechanism is available through dislocation climb.[20–22] The associated relaxation times are of the order of 50 ms. Thus pure permeation is shorted out. However, one should bear in mind that dislocation climb itself does involve local permeative flow.

5. EFFECT OF FLOW

Finally, we wish to discuss how the undulation is affected by a laminar flow parallel to the layers. The problem now becomes very complicated and intrinsically nonlinear. One can hardly venture to make any quantitative statements.

Nevertheless, let us return to eqs. (2.18–21), where now $V \neq 0$ is given by Eq. (2.15). We introduce the abbreviations

$$\beta = \frac{dV}{dz} = a_1 + 2a_2 z, \tag{5.1}$$

$$\gamma = \frac{1}{q}\frac{d^2V}{dz^2} = \frac{2a_2}{q}, \tag{5.2}$$

$$\tilde{\omega} = \omega - qV = \omega - q(a_0 + a_1 z + a_2 z^2). \tag{5.3}$$

Thus, β is a local shear rate and $\tilde{\omega}$ a kind of effective local frequency.

Repeating the reasoning that led to Eq. (3.7) we now find the differential equation:

$$\begin{aligned} i\tilde{\omega}T\frac{1}{q^4}\frac{d^4u}{dz^4} - 4i\beta T\frac{1}{q^3}\frac{d^3u}{dz^3} + \left[1 - \left(\frac{\tilde{\omega}}{Cq}\right)^2 - 2i(\tilde{\omega} + 3\gamma)T\right]\frac{1}{q^2}\frac{d^2u}{dz^2} \\ + \left[2\frac{\tilde{\omega}\beta}{(Cq)^2} + 4i\beta T\right]\frac{1}{q}\frac{du}{dz} \\ - \left[(\lambda q)^2 - \left(\frac{\tilde{\omega}}{Cq}\right)^2 - \alpha - i(\tilde{\omega} + 2\gamma)T\right]u = 0. \end{aligned} \tag{5.4}$$

Clearly, even from a formal mathematical point of view, the presence of flow has a profound effect. It introduces odd derivatives into the differential Eq. (5.4). In the physically most important case when the average shear rate does not vanish ($a_1 \neq 0$), the parity of u is destroyed. Moreover, the coefficients cease to be constant; they become, through $\tilde{\omega}$ and β, quadratic or linear functions of z.

In the short wave limit ($1/q \to 0$) we may neglect the first two terms in Eq. (5.4). Then u can be expressed in terms of parabolic cylinder functions. If one, moreover, neglects the inertial terms (containing C) then u simplifies to an Airy function. Yet, even these functions still depend on complex parameters. Their physical interpretation remains obscure, and, in view of the approximations already made, the necessary effort does not seem to be justified. An alternative approach is to treat the flow as a small perturbation.[23]

Equation (5.4) implies that in the presence of flow the ansatz Eq. (3.20) assuming a single cosine mode, is no longer admissible. Higher modes, in particular odd ones, must also be excited. Since the threshold dilation of the n-th mode is, by Eqs. (3.19 and 26) proportional to the z-component of its wave vector, $k_n = nk$, one expects to observe an increased effective threshold. For the same reason one also expects a decrease in the wavelength of the undulation.

Consider now the case of transversal flow. Again, the situation is described by Eq. (2.3). However we assume now that the flow takes place in the y-direction:

$$\mathbf{v}_0 = (0, V, 0). \tag{5.5}$$

Then, in a linear regime, the undulation and the flow become totally decoupled. Hence, an unperturbed undulation will develop in the x-direction with threshold dilation α_c and longitudinal wave vector component q_c. Yet this is possible only in the direction precisely perpendicular to the flow plane (yz). The undulation parallel to the flow develops later at some higher threshold dilation. The two together would then form a well-aligned rectangular pattern, such as, indeed, is observed.[9]

Thus the most conspicuous effect expected for a weak laminar flow is the breaking of rotational symmetry and the alignment of the undulation pattern.

The ideas put forward qualitatively in this section are the subject of a more profound quantitative study based on perturbation theory which is currently under way. So far, the results seem to confirm the present conjectures.[23]

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